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EXISTENCE OF NONOSCILLATORY SOLUTIONS TO FIRST-ORDER NEUTRAL DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT:
In this paper, we study the existence of nonoscillatory solution of first-order neutral dynamic equations with delay and advance terms on Time Scales. Some sufficient conditions for the existence of positive solutions are obtained. We use the Banach contraction principle to prove our results.
KEYWORDS:
Dynamic equations, nonoscillation, positive solution, Banach contraction principle.

I. INTRODUCTION

In this paper we consider a first-order neutral dynamic equation

$$[x(n)+P_1(n)x(n-\tau_1)+P_2(n)x(n+\tau_2)]^\Delta+Q_1(n)x(n-\sigma_1)+Q_2(n)x(n+\sigma_2)=0 \qquad (1.1)$$

where $P_1,P_2\in C([t_0,\infty),R), Q_1,Q_2\in C([t_0,\infty),[0,\infty)), \tau_1,\tau_2>0,$ and $\sigma_1,\sigma_2\geq 0.$

Let $m=\max\{\tau_1,\sigma_1\}$. We give some new criteria for the existence of non-oscillatory solutions of (1.1). Recently, the existence of non-oscillatory solutions of neutral differential equations and difference equations have been investigated by many authors see books [1,2,9,11] and papers [5,6,8,10,12,13] and the references contained therein.

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D. Thesis [7] in order to unify continuous and discrete analysis. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the reals, and scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to plenty of applications, among them the study of population dynamic models (see [3]). A book on the subject of time scales by Bohner and Peterson [3,4] summarizes and organizes much of the time scale calculus. A solution of the dynamic equation (1.1) is called eventually positive if there exists a positive integer n_0 such that $x(n)>0$ for $n\in N(n_0)$. If there exists a positive integer n_0 such that $x(n)<0$ for $n\in N(n_0)$, then (1.1) is called eventually negative.

The solution of the dynamic equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory. We need the following important theorem to prove out main results.

Theorem 1.1 (Banach's Contraction Mapping Principle). A contraction mapping on a complete metric space has exactly one fixed point.

II. MAIN RESULTS

To show that an operator S satisfies the conditions for the contraction mapping principle, we consider different cases for the ranges of the coefficients $P_1(t)$ and $P_2(t)$.

THEOREM 2.1 Assume that $0 \le P_1(n) \le p_1 < 1, 0 \le P_2(n) < p_2 < 1 - p_1$ and

$$\int_{t_0}^{\infty} Q_1(s)\Delta s < \infty, \int_{t_0}^{\infty} Q_2(s)\Delta s < \infty \qquad (2.1)$$

Then (1.1) has a bounded non-oscillatory solution.

Proof: Because of (2.1) we can choose $n_1 \ge n_0$,

$$n_1 \ge n_0 + \max\{\tau_1 + \sigma_1\} \qquad (2.2)$$

Sufficiently large such that

$$\int_{t}^{\infty} Q_1(s)\Delta s \le \frac{M_2 - \alpha}{M_2}, n \ge n_1, \qquad (2.3)$$

$$\int_{t}^{\infty} Q_2(s)\Delta s \le \frac{\alpha - (p_1 + p_2)M_2 - M_1}{M_2}, n \ge n_1 \qquad (2.4)$$

where M_1 and M_2 are positive constants such that

$(p_1 + p_2)M_2 + M_1 < M_2$ and $\alpha \in ((p_1 + p_2)M_2 + M_1, M_2)$.

Let $l_{n_0}^{\infty}$ be the set of all real sequence with the norm $\| x \| = \sup | x(n) | < \infty$. Then $l_{n_0}^{\infty}$ is a Banach space. We define a closed, bounded and convex subset Ω of $l_{n_0}^{\infty}$ as follows

$$\Omega = \{x \in l_{n_0}^{\infty} : M_1 \le x(n) \le M_2, n \ge n_0\}.$$

Define a mapping $S : \Omega \to l_{n_0}^{\infty}$ as follows

$$(Sx)(n) = \begin{cases} \alpha - P_1(n)x(n-\tau_1) - P_2(n)x(n+\tau_2) \\ + \int_{t}^{\infty} [Q_1(s)x(s-\sigma_1) - Q_2(s)x(s+\sigma_2)]\Delta s, n \ge n_1, \\ (Sx)(n_1), n_0 \le n \le n_1 \end{cases}$$

Obviously Sx is continuous. For $n \ge n_1$ and $x \in \Omega$, from (2.3) and (2.4), respectively, it follows that

$$(Sx)(n) \le \alpha + \int_{t}^{\infty} Q_1(s)x(s-\sigma_1)\Delta s$$

$$= \alpha + M_2 \int_{t}^{\infty} Q_1(s)\Delta s$$

$$= \alpha + M_2 \left(\frac{M_2 - \alpha}{M_2} \right)$$

$\therefore (Sx)(n) \le M_2$.

Furthermore we have

$$(Sx)(n) \ge \alpha - P_1(n)x(n-\tau_1) - P_2(n)x(n+\tau_2) - \int_t^\infty Q_2(s)x(s+\sigma_2)\Delta s$$

$$\ge \alpha - p_1 M_2 - p_2 M_2 - M_2 \int_t^\infty Q_2(s)\Delta s$$

$$= \alpha - p_1 M_2 - p_2 M_2 - M_2 \left(\frac{\alpha - (p_1+p_2)M_2 - M_1}{M_2} \right)$$

$\therefore (Sx)(n) \ge M_1$

Hence

$M_1 \le (Sx)(n) \le M_2$ for $n \ge n_1$

Thus we have proved that $(Sx)(n) \in \Omega$ for any $x \in \Omega$.

This mean that $S\Omega \subset \Omega$. To apply the contraction mapping principle, the remaining is to show that S is a contraction mapping on Ω. Thus $x_1, x_2 \in \Omega$ and $n \le n_1$,

$$|(Sx_1)(n) - (Sx_2)(n)|$$

$$= \Big|\alpha - P_1(n)x_1(n-\tau_1) - P_2(n)x_1(n+\tau_2) + \int_t^\infty [Q_1(s)x_1(s-\sigma_1) - Q_2(s)x_1(s+\sigma_2)]\Delta s$$

$$-(\alpha - P_1(n)x_2(n-\tau_1) - P_2(n)x_2(n+\tau_2) + \int_t^\infty [Q_1(s)x_2(s-\sigma_1) - Q_2(s)x_2(s+\sigma_2)]\Delta s)\Big|$$

$$\le P_1(n)|x_1(n-\tau_1) - x_2(n-\tau_1)| + P_2(n)|x_1(n+\tau_2) - x_2(n+\tau_2)|$$

$$+\int_t^\infty Q_1(s)|x_1(s-\sigma_1) - x_2(s-\sigma_1)|\Delta s + \int_t^\infty Q_2(s)|x_1(s+\sigma_1) - x_2(s+\sigma_2)|\Delta s$$

$$\le p_1\|x_1 - x_2\| + p_2\|x_1 - x_2\| + \int_t^\infty Q_1(s)\|x_1 - x_2\|\Delta s + \int_t^\infty Q_2(s)\|x_1 - x_2\|\Delta s$$

$$= \left(p_1 + p_2 + \int_t^\infty Q_1(s)\Delta s + \int_t^\infty Q_2(s)\Delta s \right)\|x_1 - x_2\|$$

$$= \left(p_1 + p_2 + \frac{M_2 - \alpha}{M_2} + \frac{\alpha - (p_1+p_2)M_2 - M_1}{M_2} \right)\|x_1 - x_2\|$$

$$= \frac{M_2 - M_1}{M_2}\left(\|x_1 - x_2\|\right)$$

$=\lambda_1\|x_1-x_2\|$

Where $\lambda_1=1-\dfrac{M_1}{M_2}$. This implies that

$\|(Sx_1)(n)-(Sx_2)(n)\|\le\lambda_1\|x_1-x_2\|$

Thus we have to proved that S is a contraction mapping on Ω. In fact $x_1,x_2\in\Omega$ and $n\ge n_1$ we have

$|(Sx_1)(n)-(Sx_2)(n)|\le p(n)|x_1(n-\tau_1)-x_2(n-\tau_1)|\le\lambda_1\|x_1-x_2\|$

Since $0<\lambda_1<1$. We conclude that S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof.

THEOREM 2.2. Assume that $0\le P_1(n)\le p_1<1, p_1-1<p_2\le P_2(n)\le 0$ and (2.1) hold, then (1.1) has a bounded non-oscillatory solution.

Proof: Because of (2.1),we can choose $n_1\ge n_0$ sufficiently large satisfying (2.2) such that

$$\int_t^\infty Q_1(s)\Delta s\le\frac{(1+p_2)N_2-\alpha}{N_2}, n\ge n_1, \tag{2.5}$$

$$\int_t^\infty Q_2(s)\Delta s\le\frac{\alpha-p_1N_2-N_1}{N_2}, n\ge n_1 \tag{2.6}$$

Where N_1 and N_2 are positive constants such that

$N_1+p_1N_2<(1+p_2)N_2$ and $\alpha\in(N_1+p_1N_2,(1+p_2)N_2)$.

Let $l_{n_0}^\infty$ be the set of all real sequence with the norm $\|x\|=\sup|x(n)|<\infty$. Then $l_{n_0}^\infty$ is a Banach space. We define a closed, bounded and convex subset Ω of $l_{n_0}^\infty$ as follows

$\Omega=\{x\in l_{n_0}^\infty : N_1\le x(n)\le N_2, n\ge n_0\}$.

Define a mapping $S:\Omega\to l_{n_0}^\infty$ as follows

$$(Sx)(n)=\begin{cases}\alpha-P_1(n)x(n-\tau_1)-P_2(n)x(n+\tau_2)\\ +\int_t^\infty[Q_1(s)x(s-\sigma_1)-Q_2(s)x(s+\sigma_2)]\Delta s, n\ge n_1,\\ (Sx)(n_1), n_0\le n\le n_1.\end{cases}$$

Obviously Sx is continuous. For $n\ge n_1$, and $x\in\Omega$, from (2.5) and (2.6), respectively, it follows that

$$(Sx)(n)\le\alpha-P_2(n)x(n+\tau_2)+\int_t^\infty Q_1(s)x(s-\sigma_1)\Delta s$$

$$\leq \alpha - p_2 N_2 + N_2 \int_t^\infty Q_1(s)\Delta s$$

$$= \alpha - p_2 N_2 + N_2 \left(\frac{(1+p_2)N_2 - \alpha}{N_2} \right)$$

$\therefore (Sx)(n) \leq N_2$

Furthermore we have

$$(Sx)(n) \geq \alpha - P_1(n)x(n-\tau_1) - \int_t^\infty Q_2(s)x(s+\sigma_2)\Delta s$$

$$\geq \alpha - p_1 N_2 - N_2 \int_t^\infty Q_2(s)\Delta s$$

$$= \alpha - p_1 N_2 - N_2 \left(\frac{\alpha - p_1 N_2 - N_1}{N_2} \right)$$

$\therefore (Sx)(n) \geq N_1$

Hence

$N_1 \leq (Sx)(n) \leq N_2$ for $n \geq n_1$

Thus we have proved that $(Sx)(n) \in \Omega$ for any $x \in \Omega$.

This mean that $S\Omega \subset \Omega$. To apply the contraction mapping principle, the remaining is to show that S is a contraction mapping on Ω. Thus $x_1, x_2 \in \Omega$ and $n \leq n_1$,

$$|(Sx_1)(n) - (Sx_2)(n)|$$

$$= \left| \alpha - P_1(n)x_1(n-\tau_1) - P_2(n)x_1(n+\tau_2) + \int_t^\infty [Q_1(s)x_1(s-\sigma_1) - Q_2(s)x_1(s+\sigma_2)]\Delta s \right.$$

$$\left. -(\alpha - P_1(n)x_2(n-\tau_1) - P_2(n)x_2(n+\tau_2) + \int_t^\infty [Q_1(s)x_2(s-\sigma_1) - Q_2(s)x_2(s+\sigma_2)]\Delta s) \right|$$

$$\leq P_1(n)|x_1(n-\tau_1) - x_2(n-\tau_1)| + P_2(n)|x_1(n+\tau_2) - x_2(n+\tau_2)|$$
$$+ \int_t^\infty Q_1(s)|x_1(s-\sigma_1) - x_2(s-\sigma_1)|\Delta s + \int_t^\infty Q_2(s)|x_1(s+\sigma_2) - x_2(s+\sigma_2)|\Delta s$$

$$\leq p_1\|x_1 - x_2\| - p_2\|x_1 - x_2\| + \int_t^\infty Q_1(s)\|x_1 - x_2\|\Delta s + \int_t^\infty Q_2(s)\|x_1 - x_2\|\Delta s$$

$$= \left(p_1 - p_2 + \int_t^\infty Q_1(s)\Delta s + \int_t^\infty Q_2(s)\Delta s \right) \|x_1 - x_2\|$$

$$=\left(p_1-p_2+\frac{(1+p_2)N_2-\alpha}{N_2}+\frac{\alpha-p_1N_2-N_1}{N_2}\right)\|x_1-x_2\|$$

$$=\frac{N_2-N_1}{N_2}\left(\|x_1-x_2\|\right)$$

$$=\lambda_2\|x_1-x_2\|$$

Where $\lambda_2=1-\frac{N_1}{N_2}$. This implies that

$$\|(Sx_1)(n)-(Sx_2)(n)\|\le\lambda_2\|x_1-x_2\|$$

Thus we have to proved that S is a contraction mapping on Ω. In fact $x_1,x_2\in\Omega$ and $n\ge n_1$ we have

$$|(Sx_1)(n)-(Sx_2)(n)|\le p(n)|x_1(n-\tau_1)-x_2(n-\tau_1)|\le\lambda_2\|x_1-x_2\|$$

Since $0<\lambda_2<1$. We conclude that S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof.

Theorem 2.3. Assume that $1<p_1\le P_1(n)<p_{1_0}<\infty, 0\le P_2(n)\le p_2<p_1-1$ and (2.1) hold, then (1.1) has a bounded non-oscillatory solution.

Proof: In view of (2.1), we can choose $n_1\ge n_0$

$$n_1+\tau_1\ge n_0+\sigma_1, \tag{2.7}$$

Sufficiently large such that

$$\int_t^\infty Q_1(s)\Delta s\le\frac{p_1M_4-\alpha}{M_4}, n\ge n_1, \tag{2.8}$$

$$\int_t^\infty Q_2(s)\Delta s\le\frac{\alpha-p_{1_0}M_3-(1+p_2)M_4}{M_4}, n\ge n_1 \tag{2.9}$$

Where M_3 and M_4 are positive constants such that

$$p_{1_0}M_3+(1+p_2)M_4<p_1M_4 \text{ and } \alpha\in(p_{1_0}M_3+(1+p_2)M_4, p_1M_4).$$

Let $l_{n_0}^\infty$ be the set of all real sequence with the norm $\|x\|=\sup|x(n)|<\infty$. Then $l_{n_0}^\infty$ is a Banach space. We define a closed, bounded and convex subset Ω of $l_{n_0}^\infty$ as follows

$$\Omega=\{x\in l_{n_0}^\infty : M_3\le x(n)\le M_4, n\ge n_0\}.$$

Define a mapping $S:\Omega\to l_{n_0}^\infty$ as follows

$$(Sx)(n)=\begin{cases}\dfrac{1}{P_1(n+\tau_1)}\{\alpha-x(n+\tau_1)-P_2(n+\tau_1)x(n+\tau_1+\tau_2)\\ +\displaystyle\int_{t+\tau_1}^{\infty}[Q_1(s)x(s-\sigma_1)-Q_2(s)x(s+\sigma_2)]\Delta s\}, n\geq n_1,\\ (Sx)(n_1), n_0\leq n\leq n_1.\end{cases}$$

Clearly Sx is continuous. For $n\geq n_{1'}$ and $x\in\Omega$, from (2.8) and (2.9), respectively, it follows that

$$(Sx)(n)\leq\frac{1}{P_1(n+\tau_1)}\left(\alpha+\int_{t+\tau_1}^{\infty}Q_1(s)x(s-\sigma_1)\Delta s\right)$$

$$\leq\frac{1}{p_1}\left(\alpha+M_4\int_t^{\infty}Q_1(s)\Delta s\right)$$

$$=\frac{1}{p_1}\left(\alpha+M_4\left(\frac{p_1M_4-\alpha}{M_4}\right)\right)$$

$$\therefore (Sx)(n)\leq M_4$$

Furthermore we have

$$(Sx)(n)\geq\frac{1}{P_1(n+\tau_1)}(\alpha-x(n+\tau_1)-P_2(n+\tau_1)x(n+\tau_1+\tau_2)-\int_{t+\tau_1}^{\infty}Q_2(s)x(s+\sigma_2)\Delta s)$$

$$\geq\frac{1}{P_1(n+\tau_1)}\left(\alpha-M_4-p_2M_4-M_4\int_t^{\infty}Q_2(s)\Delta s\right)$$

$$\geq\frac{1}{p_{1_0}}\left(\alpha-(1+p_2)M_4-M_4\int_t^{\infty}Q_2(s)\Delta s\right)$$

$$=\frac{1}{p_{1_0}}\left(\alpha-(1+p_2)M_4-M_4\left(\frac{\alpha-p_{1_0}M_{3-}(1+p_2)M_4}{M_4}\right)\right)$$

$$(Sx)(n)\geq M_3$$

Hence

$$M_3\leq(Sx)(n)\leq M_4 \text{ for } n\geq n_1$$

This we have proved that $(Sx)(n)\in\Omega$ for any $x\in\Omega$.

This mean that $S\Omega\subset\Omega$. To apply the contraction mapping principle, the remaining is to show that S is a contraction mapping on Ω. Thus $x_1,x_2\in\Omega$ and $n\leq n_1$,

$$
\begin{aligned}
&|(Sx_1)(n)-(Sx_2)(n)| \\
&=\left|\frac{1}{P_1(n+\tau_1)}\{\alpha - x_1(n+\tau_1)-P_2(n+\tau_1)x_1(n+\tau_1+\tau_2)+\int_{t+\tau_1}^{\infty}[Q_1(s)x_1(s-\sigma_1)-Q_2(s)x_1(s+\sigma_2)]\Delta s\}\right. \\
&\left.-\left(\frac{1}{P_1(n+\tau_1)}\{\alpha - x_2(n+\tau_1)-P_2(n+\tau_1)x_2(n+\tau_1+\tau_2)+\int_{t+\tau_1}^{\infty}[Q_1(s)x_2(s-\sigma_1)-Q_2(s)x_2(s+\sigma_2)]\Delta s\}\right)\right| \\
&\le \frac{1}{P_1(n+\tau_1)}(|x_1(n+\tau_1)-x_2(n+\tau_1)|+P_2(n+\tau_1)|x_1(n+\tau_1+\tau_2)-x_2(n+\tau_1+\tau_2)| \\
&+\int_{t+\tau_1}^{\infty}Q_1(s)|x_1(s-\sigma_1)-x_2(s-\sigma_1)|\Delta s+\int_{t+\tau_1}^{\infty}Q_2(s)|x_1(s+\sigma_2)-x_2(s+\sigma_2)|\Delta s) \\
&\le \frac{1}{p_1}\left(\|x_1-x_2\|+p_2\|x_1-x_2\|+\int_t^{\infty}Q_1(s)\|x_1-x_2\|\Delta s+\int_t^{\infty}Q_2(s)\|x_1-x_2\|\Delta s\right) \\
&=\frac{1}{p_1}\left(1+p_2+\int_t^{\infty}Q_1(s)\Delta s+\int_t^{\infty}Q_2(s)\Delta s\right)\|x_1-x_2\| \\
&=\frac{1}{p_1}\left(1+p_2+\frac{p_1M_4-\alpha}{M_4}+\frac{\alpha-p_{1_0}M_3-(1+p_2)M_4}{M_4}\right)\|x_1-x_2\| \\
&=\frac{1}{p_1}\left(\frac{p_1M_4-p_{1_0}M_3}{M_4}\right)\|x_1-x_2\| \\
&=\lambda_3\|x_1-x_2\|
\end{aligned}
$$

Where $\lambda_3 = 1-\dfrac{p_{1_0}M_3}{p_1M_4}$ This implies that

$$\|(Sx_1)(n)-(Sx_2)(n)\| \le \lambda_3\|x_1-x_2\|$$

Thus we have to proved that S is a contraction mapping on Ω. In fact $x_1, x_2 \in \Omega$ and $n \ge n_1$ we have

$$|(Sx_1)(n)-(Sx_2)(n)| \le p(n)|x_1(n-\tau_1)-x_2(n-\tau_1)| \le \lambda_3\|x_1-x_2\|$$

Since $0<\lambda_3<1$. We conclude that S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof.

THEOREM 2.4. Assume that $1<p_1 \le P_1(n) < p_{1_0} < \infty, 1-p_1 < p_2 \le P_2(n) \le 0$ and (2.1) hold, then (1.1) has a bounded non-oscillatory solution.

Proof: In view of (2.1), we can choose $n_1 \ge n_0$ sufficiently large satisfying (2.7) such that

$$\int_t^{\infty}Q_1(s)\Delta s \le \frac{(p_1+p_2)N_4-\alpha}{N_4}, n \ge n_1, \tag{2.10}$$

$$\int_t^\infty Q_2(s)\Delta s \le \frac{\alpha - p_{1_0}N_3 - N_4}{N_4}, n \ge n_1, \ (2.11)$$

Where N_3 and N_4 are positive constants such that

$p_{1_0}N_3 + N_4 < (p_1+p_2)N_4$ and $\alpha \in (p_{1_0}N_3 + N_4, (p_1+p_2)N_4)$.

Let $l_{n_0}^\infty$ be the set of all real sequence with the norm $\|x\| = \sup|x(n)| < \infty$. Then $l_{n_0}^\infty$ is a Banach space. We define a closed, bounded and convex subset Ω of $l_{n_0}^\infty$ as follows

$$\Omega = \{x \in l_{n_0}^\infty : N_3 \le x(n) \le N_4, n \ge n_0\}.$$

Define a mapping $S:\Omega \to l_{n_0}^\infty$ as follows

$$(Sx)(n) = \begin{cases} \dfrac{1}{P_1(n+\tau_1)}\{\alpha - x(n+\tau_1) - P_2(n+\tau_1)x(n+\tau_1+\tau_2) \\ + \displaystyle\int_{t+\tau_1}^\infty [Q_1(s)x(s-\sigma_1) - Q_2(s)x(s+\sigma_2)]\Delta s\}, n \ge n_1, \\ (Sx)(n_1), n_0 \le n \le n_1. \end{cases}$$

Clearly Sx is continuous. For $n \ge n_{1'}$ and $x \in \Omega$, from (2.10) and (2.11), respectively, it follows that

$$(Sx)(n) \le \frac{1}{P_1(n+\tau_1)}\left(\alpha - P_2(n+\tau_1)x(n+\tau_1+\tau_2) + \int_{t+\tau_1}^\infty Q_1(s)x(s-\sigma_1)\Delta s\right)$$

$$\le \frac{1}{p_1}\left(\alpha - p_2N_4 + N_4\int_t^\infty Q_1(s)\Delta s\right)$$

$$= \frac{1}{p_1}\left(\alpha - p_2N_4 + N_4\left(\frac{(p_1+p_2)N_4 - \alpha}{N_4}\right)\right)$$

$$\therefore (Sx)(n) \le N_4$$

Furthermore we have

$$(Sx)(n) \ge \frac{1}{P_1(n+\tau_1)}\left(\alpha - x(n+\tau_1) - \int_{t+\tau_1}^\infty Q_2(s)x(s+\sigma_2)\Delta s\right)$$

$$\ge \frac{1}{p_{1_0}}\left(\alpha - N_4 - N_4\int_t^\infty Q_2(s)\Delta s\right)$$

$$= \frac{1}{p_{1_0}}\left(\alpha - N_4 - N_4\left(\frac{\alpha - p_{1_0}N_3 - N_4}{N_4}\right)\right)$$

$\therefore (Sx)(n) \geq N_3$

Hence

$N_3 \leq (Sx)(n) \leq N_4$ for $n \geq n_1$

This we have proved that $(Sx)(n) \in \Omega$ for any $x \in \Omega$.

This mean that $S\Omega \subset \Omega$. To apply the contraction mapping principle, the remaining is to show that S is a contraction mapping on Ω. Thus $x_1, x_2 \in \Omega$ and $n \leq n_1$,

$$|(Sx_1)(n) - (Sx_2)(n)|$$

$$= \left|\frac{1}{P_1(n+\tau_1)}\{\alpha - x_1(n+\tau_1) - P_2(n+\tau_1)x_1(n+\tau_1+\tau_2) + \int_{t+\tau_1}^{\infty}[Q_1(s)x_1(s-\sigma_1) - Q_2(s)x_1(s+\sigma_2)]\Delta s\}\right.$$

$$\left.-\left(\frac{1}{P_1(n+\tau_1)}\{\alpha - x_2(n+\tau_1) - P_2(n+\tau_1)x_2(n+\tau_1+\tau_2) + \int_{t+\tau_1}^{\infty}[Q_1(s)x_2(s-\sigma_1) - Q_2(s)x_2(s+\sigma_2)]\Delta s\}\right)\right|$$

$$\leq \frac{1}{P_1(n+\tau_1)}(|x_1(n+\tau_1) - x_2(n+\tau_1)| + P_2(n+\tau_1)|x_1(n+\tau_1+\tau_2) - x_2(n+\tau_1+\tau_2)|$$

$$+ \int_{t+\tau_1}^{\infty} Q_1(s)|x_1(s-\sigma_1) - x_2(s-\sigma_1)|\Delta s + \int_{t+\tau_1}^{\infty} Q_2(s)|x_1(s+\sigma_2) - x_2(s+\sigma_2)|\Delta s)$$

$$\leq \frac{1}{p_1}\left(\|x_1 - x_2\| - p_2\|x_1 - x_2\| + \int_t^{\infty} Q_1(s)\|x_1 - x_2\|\Delta s + \int_t^{\infty} Q_2(s)\|x_1 - x_2\|\Delta s\right)$$

$$= \frac{1}{p_1}\left(1 - p_2 + \int_t^{\infty} Q_1(s)\Delta s + \int_t^{\infty} Q_2(s)\Delta s\right)\|x_1 - x_2\|$$

$$= \frac{1}{p_1}\left(1 - p_2 + \frac{(p_1+p_2)N_4 - \alpha}{N_4} + \frac{\alpha - p_{1_0}N_3 - N_4}{N_4}\right)\|x_1 - x_2\|$$

$$= \frac{1}{p_1}\left(\frac{p_1N_4 - p_{1_0}N_3}{N_4}\right)\|x_1 - x_2\|$$

$$= \lambda_4\|x_1 - x_2\|$$

where $\lambda_4 = 1 - \frac{p_{1_0}N_3}{p_1N_4}$. This implies that

$$\|(Sx_1)(n) - (Sx_2)(n)\| \leq \lambda_4\|x_1 - x_2\|$$

Thus we have to proved that S is a contraction mapping on Ω. In fact $x_1, x_2 \in \Omega$ and $n \geq n_1$ we have

$$|(Sx_1)(n)-(Sx_2)(n)| \le p(n)|x_1(n-\tau_1)-x_2(n-\tau_1)| \le \lambda_4 \|x_1-x_2\|$$

Since $0<\lambda_4<1$. We conclude that S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof.

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